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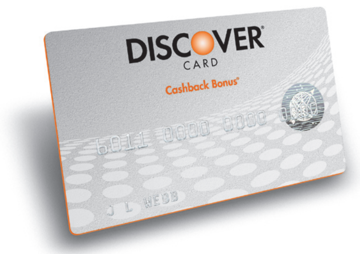

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C H A P T E R 4

The Inverse Trigonometric Functions

In previous chapters we have used what we called the inverse trigonometric functions. For example, we stated that if $\sin \theta = 0.3$, then one value of θ is $\theta = \sin^{-1} 0.3$. In this chapter we examine these inverse trigonometric functions in complete detail.

We use these functions both to solve trigonometric equations like that in the last paragraph, to solve triangles, and to describe angles in terms of given information. These functions are also useful in advanced mathematics (calculus in particular) where they provide a means to simplify certain algebraic expressions.

We begin the chapter by examining the general topic of inverse functions, and then apply this topic to trigonometry.

4–1 The inverse of a function

In section 2–1 we introduced the idea of the inverse of a function. We stated that

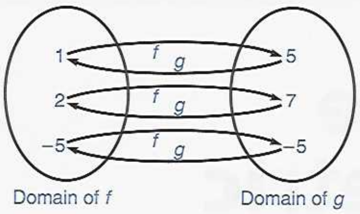
- A function is a set of ordered pairs in which no first element is repeated.
- A one-to-one function is a function in which no second element is repeated.
- Reversing the elements of the ordered pairs of a function produces a function if, and only if, the function is one to one.

For example,

$$H = \{(1,5), (2,7), (-5,-5)\}$$

is a one-to-one function, and its inverse function is

$$H^{-1} = \{(5,1), (7,2), (-5,-5)\}$$



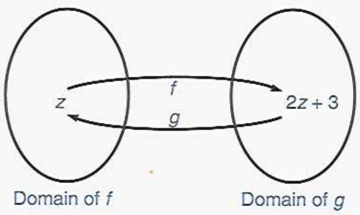


Figure 4–1

Showing that two given functions are inverses

Now consider the two functions $f(x) = 2x + 3$ and $g(x) = \dfrac{x-3}{2}$. By computation we could determine the following facts.

$$f(1) = 5 \text{ and } g(5) = 1$$
$$f(2) = 7 \text{ and } g(7) = 2$$
$$f(-5) = -7 \text{ and } g(-7) = -5$$

(Compare to H and H^{-1} on page 133 and figure 4–1.)

Whatever value z we try, f sends z to some value z', and g sends z' back to z. In fact we can prove this; let z represent any real number. Then,

$$f(z) = 2z + 3 \text{ and}$$
$$g(2z+3) = \frac{(2z+3)-3}{2} = z$$

Also

$$g(z) = \frac{z-3}{2} \text{ and}$$
$$f\left(\frac{z-3}{2}\right) = 2\left(\frac{z-3}{2}\right) + 3 = z$$

When two functions f and g act this way we say they are inverse functions. This is because, whenever an ordered pair (a,b) is in f, the ordered pair formed by reversing its elements (b,a) is in g. The functions H and H^{-1}, and f and g, above, are examples of this.

The notation for the inverse of a function is the superscript -1, so using this notation, if $f(x) = 2x + 3$ (as above) we can say that $f^{-1}(x) = \dfrac{x-3}{2}$. Note the superscript -1, when applied to the name of a function, is *not* an exponent; it does not indicate division, as it does if applied as an exponent of a real valued expression.

$$f^{-1}(x) \text{ does not mean } \frac{1}{f(x)}$$

To show that two functions *f* and *g* are inverses of each other

Show that

[1] If $f(x) = y$, then $g(y) = x$, and
[2] If $g(x) = y$, then $f(y) = x$.

Note In practice "y" represents an expression in x.

■ ***Example 4–1 A***

Show that f and g are the inverse functions of each other.

1. $f(x) = \frac{1}{3}x - 1$; $g(x) = 3x + 3$

[1] Assume $f(x) = y$, then $y = \frac{1}{3}x - 1$. — Replace $f(x)$ by y

Show that $g(y) = x$:

$$\begin{aligned} g(y) &= 3y + 3 && \text{Replace } x \text{ by } y \text{ in } g(x) = 3x + 3 \\ &= 3(\tfrac{1}{3}x - 1) + 3 && \text{Replace } y \text{ by } \tfrac{1}{3}x - 1 \\ &= x - 3 + 3 \\ &= x \end{aligned}$$

[2] Assume $g(x) = y$, then $y = 3x + 3$.

Show that $f(y) = x$:

$$\begin{aligned} f(y) &= \tfrac{1}{3}y - 1 && \text{Replace } x \text{ by } y \text{ in } f(x) = \tfrac{1}{3}x - 1 \\ &= \tfrac{1}{3}(3x + 3) - 1 && \text{Replace } y \text{ by } 3x + 3 \\ &= x + 1 - 1 \\ &= x \end{aligned}$$

Thus, we have shown conditions [1] and [2] above, so f and g are inverse functions.

2. $f(x) = \sqrt{x}$; $g(x) = x^2$, $x \geq 0$

[1] $y = \sqrt{x}$ — Let y represent $f(x)$

$$\begin{aligned} g(y) &= y^2 && \text{Determine } g(y) \\ &= (\sqrt{x})^2 \\ &= x && (\sqrt{a})^2 = a \end{aligned}$$

[2] $y = x^2$, $x \geq 0$ — Let y represent $g(x)$

$$\begin{aligned} f(y) &= \sqrt{y} && \text{Determine } f(y) \\ &= \sqrt{x^2} \\ &= x && \sqrt{a^2} = a \text{ if } a \geq 0 \end{aligned}$$

Thus, we have shown conditions [1] and [2], so f and g are inverse functions. ■

Graphical analysis of relations for the function and one-to-one properties

The graph of a relation can be used to determine whether that relation is a function, and whether or not a function is one to one. This is done by the **vertical line test** and the **horizontal line test.**

> **Vertical line test for a function**
> If no vertical line crosses the graph of a relation in more than one place, the relation is a function.

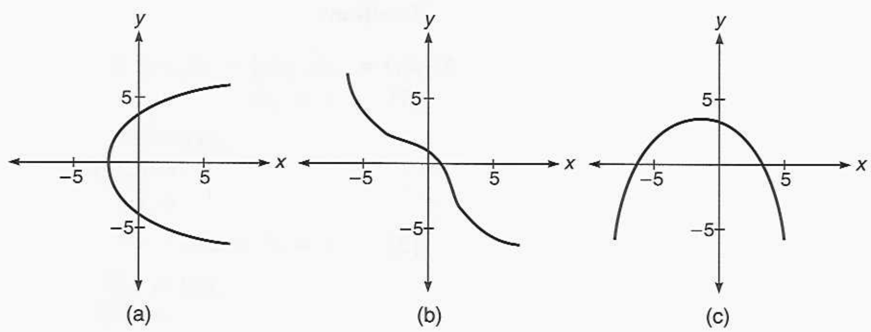
The vertical line test works for the following reason. Assume a vertical line crosses a graph at more than one point. Since these two points are in a vertical line their first components (the x-values) are equal. Therefore, the function must have two points in which the first element repeats, and it is therefore not a function.

> **Horizontal line test for a one-to-one function**
> If no horizontal line crosses the graph of a function in more than one place, the function is one to one.

The horizontal line test works for reasons similar to those for the vertical line test. If a horizontal line crosses a function at two (or more) points, then these are different domain elements (first components) with the same range elements (second component). Therefore the function is not one to one.

■ ***Example 4–1 B***

Tell which relations are functions, and which functions are one to one.



1. Relation (a) is not a function since there are clearly many vertical lines that would intersect the graph in at least two places.
2. Relation (b) is a function since no vertical line will intersect the graph in more than one place. It is also one to one since no horizontal line will intersect the graph in more than one place.
3. Relation (c) is a function by the vertical line test, but not a one-to-one function. ■

The graph of a function's inverse

The fact that the ordered pairs reverse in a function's inverse function means that *the graph of f^{-1} is a reflection of the graph of f about the line $y = x$.* By way of example, observe the graphs of the functions in example 4–1 A. These are shown in figure 4–2. To draw a graph that is symmetric about the line $y = x$ to a given graph, we draw lines perpendicular to the line $y = x$, as shown, and plot points at equal distances from this line, but on the other side of this line. Since the ordered pairs of f all reverse in f^{-1} *the domain of f is the range of f^{-1}, and the range of f is the domain of f^{-1}.*

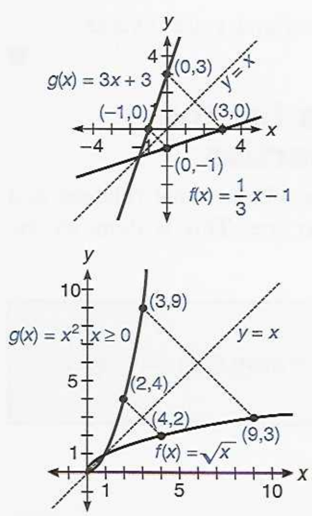
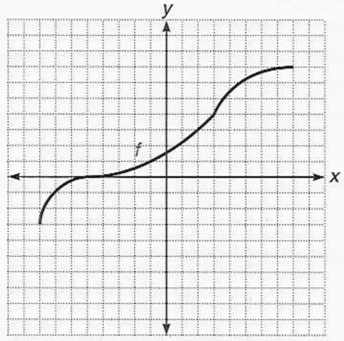


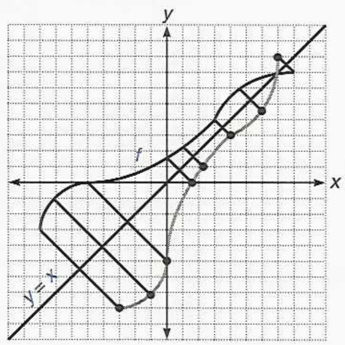
Figure 4–2

■ ***Example 4–1 C***

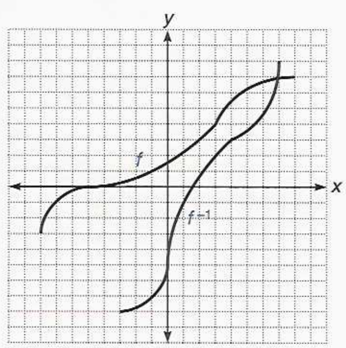
Given the graph of the one-to-one function f, sketch the graph of its inverse function f^{-1}.



To graph f^{-1} we construct the line $y = x$, which is a straight line passing through the origin with a slope of 1. We then construct various straight lines perpendicular to this line, starting on the graph of the function, and extending an equal distance to the other side of the line $y = x$.



Connecting the points that result from this process gives the graph of f^{-1}.



■

Mastery points

Can you

- Demonstrate that two functions are inverses of each other?
- Given a graph that represents a one-to-one function, graph its inverse function?

Exercise 4–1

Show that the following functions f and g are inverses of each other. Assume the domains as indicated are correct.

1. $f(x) = 2x - 7$; $g(x) = \frac{1}{2}x + 3\frac{1}{2}$

2. $f(x) = -\frac{1}{3}x + \frac{1}{2}$; $g(x) = -3x + \frac{3}{2}$

3. $f(x) = \frac{1}{3}x + \frac{8}{3}$; $g(x) = 3x - 8$

4. $f(x) = x - 1$; $g(x) = x + 1$

5. $f(x) = 2x - 5$; $g(x) = \frac{1}{2}(x + 5)$

6. $f(x) = \dfrac{x}{5} - 3$; $g(x) = 5(x + 3)$

7. $f(x) = \dfrac{2}{x - 3}$; $g(x) = \dfrac{2}{x} + 3$

8. $f(x) = \dfrac{3}{x + 2}$; $g(x) = \dfrac{3}{x} - 2$

9. $f(x) = 7 - \dfrac{3}{x}$; $g(x) = \dfrac{3}{7 - x}$

10. $f(x) = 5 + \dfrac{3}{x - 1}$; $g(x) = \dfrac{3}{x - 5} + 1$

11. $f(x) = \dfrac{x}{x - 1}$; $g(x) = \dfrac{x}{x - 1}$

12. $f(x) = \dfrac{x + 1}{x - 5}$; $g(x) = \dfrac{5x + 1}{x - 1}$

13. $f(x) = x^2 - 9, x \geq 0$; $g(x) = \sqrt{x + 9}$

14. $f(x) = \sqrt{4 - 2x}$; $g(x) = 2 - \frac{1}{2}x^2, x \geq 0$

15. $f(x) = x^3$; $g(x) = \sqrt[3]{x}$

16. $f(x) = x^3 - 3$; $g(x) = \sqrt[3]{x + 3}$

17. $f(x) = x^2 - 2x + 3, x \geq 1$; $g(x) = \sqrt{x - 2} + 1$

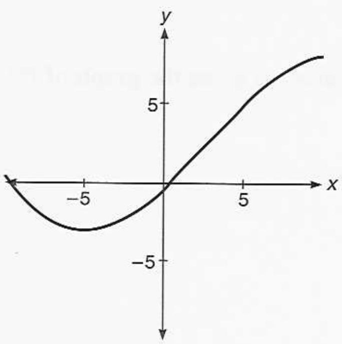
18. $f(x) = \sqrt{x + 9} - 2$; $g(x) = x^2 + 4x - 5, x \geq -2$

19. $f(x) = \dfrac{2x}{x - 3}$; $g(x) = \dfrac{3x}{x - 2}$

20. $f(x) = \dfrac{x - 3}{x - 2}$; $g(x) = 2 - \dfrac{1}{x - 1}$

Tell which relation is a function, and which functions are one to one.

21.



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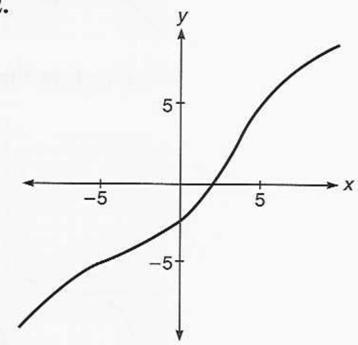

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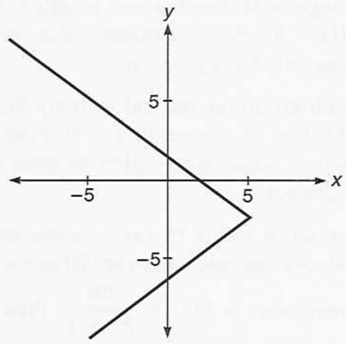

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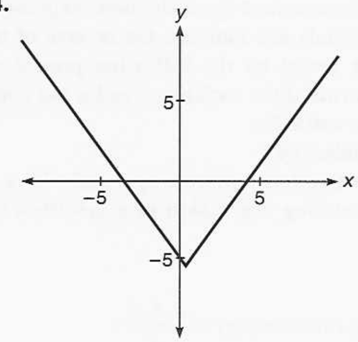
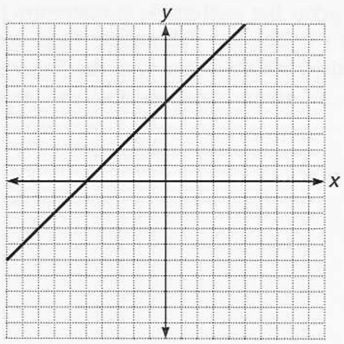


Sketch the graph of f^{-1}, given the graph of the one-to-one function *f*.

25.



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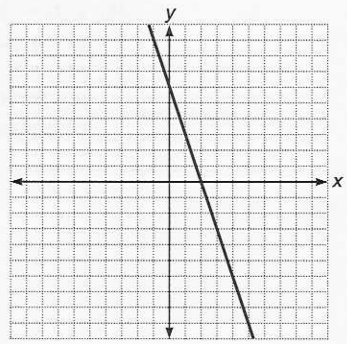

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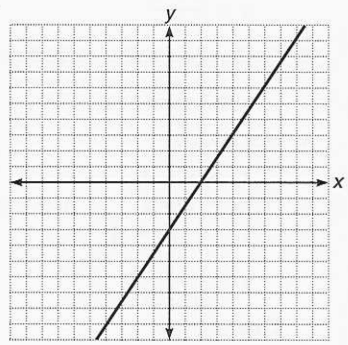

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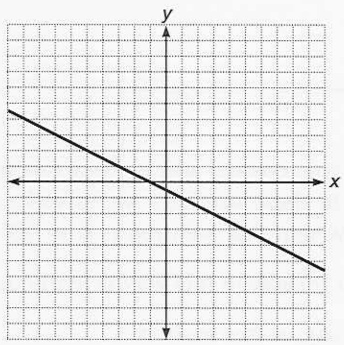

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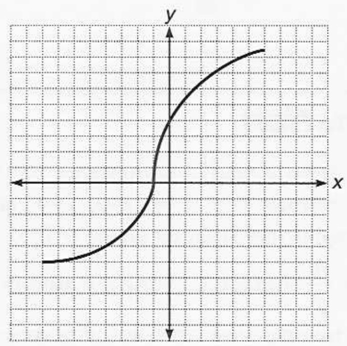

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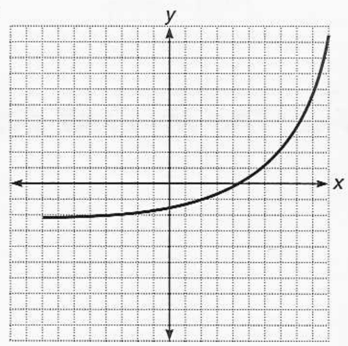

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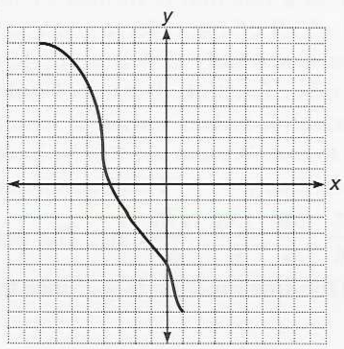

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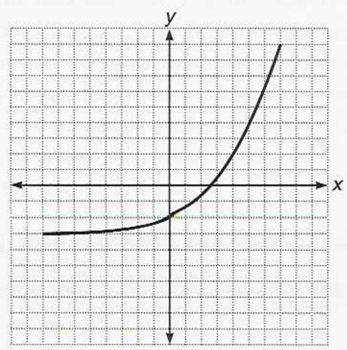


When a function is described by an algebraic expression involving polynomials and radicals, the inverse of the function can often be found by the following procedure, which is described in terms of the variables x and y, but could be in terms of any two variables:

1. Replace the $f(x)$ symbol by y.
2. Exchange the two variables; replace x by y and y by x.
3. Solve for y. The resulting expression in x describes the inverse function.

Example

Find the inverse of the function $f(x) = \dfrac{x + 1}{x}$.

1. Replace the $f(x)$ symbol by y:

$$y = \frac{x + 1}{x}$$

2. Exchange x and y variables:

$$x = \frac{y + 1}{y}$$

3. Solve for y:

$xy = y + 1$

$xy - y = 1$ All y terms on one side

$y(x - 1) = 1$

$y = \dfrac{1}{x - 1}$

This is f^{-1}:

$$f^{-1}(x) = \frac{1}{x - 1}$$

33. The area of a rectangle with width 4 and length $x + 4$, $x \geq 0$, is $A(x) = 4(x + 4)$. Find the inverse of A, which would give the value of x for a given area.

$\boxed{34.}$ A falling object with no initial vertical velocity falls a distance $d(t) = 16t^2$ feet in t seconds, $t > 0$. Find the inverse of this function, which would give the time necessary to fall a distance a.

35. In an electronic circuit in which two resistances are in parallel, and the value of the resistances are 20 ohms and x ohms, the total resistance is $R(x) = \dfrac{20x}{20 + x}$. Find the inverse of this function, which would give the value of x required for a total resistance R.

36. $C(t) = \frac{5}{9}(t - 32)$ gives the centigrade temperature for a given temperature t in degrees Fahrenheit. Find the inverse of this function, which would find the Fahrenheit temperature for a given temperature in degrees centigrade.

4–2 *The inverse sine function*

We have already encountered the inverse sine, cosine, and tangent functions in sections 1–3, 1–4, 2–3, 2–4, and 2–6. In these sections we solved for θ in equations such as $\sin\theta = 0.5$. We know that one solution is $\theta = 30°$, or $\theta = \dfrac{\pi}{6}$ (radians). Using the sine function in reverse in this way is really using the inverse sine function, which we will study in this section.

In section 2–1 we said that the inverse of a function is formed by interchanging the first and second components of all of the ordered pairs in the function and that a function has an inverse if and only if it is a one-to-one function. Since the sine function, like any other function, is a set of ordered pairs, we may ask if it has an inverse. Naturally, it does *only* if it is one to one. The sine function is, however, not one to one. We can see this by looking at its graph in figure 4–3. Note the horizontal line $y = \frac{1}{2}$. This line intersects with (crosses) the graph at many points: $\dfrac{\pi}{6}, \dfrac{5\pi}{6}, \dfrac{13\pi}{6}, \dfrac{17\pi}{6}$, etc. The points of intersection show all of the places where the sine function is $\frac{1}{2}$. Since there

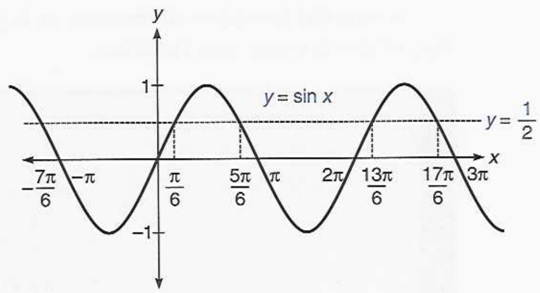


Figure 4–3

is more than one such point, there is more than one ordered pair in the sine function that has $\frac{1}{2}$ as its second element. Some of these ordered pairs are $\left(\frac{\pi}{6},\frac{1}{2}\right)$, $\left(\frac{5\pi}{6},\frac{1}{2}\right)$, $\left(\frac{13\pi}{6},\frac{1}{2}\right)$, etc. Thus, there is a repetition in the second element of these ordered pairs, and the sine function is, therefore, not one to one.

In general, if we can find a horizontal line that intersects the graph of a function in more than one point, then the function cannot be one to one. This graphic test is the horizontal line test, presented in section 4–1.

Since we can quickly see from our knowledge of the graphs of the six trigonometric functions (see chapter 3) that they would all fail the horizontal line test, we know that none of these functions is one to one.

Experience has shown, however, that there is a real need for inverses of these functions. By compromising a little, we can indeed define inverse functions for the six trigonometric functions. The compromise we must make is to form the inverse of only a small part of each function. This part is chosen so that it will include the entire range of the given function but will be one to one. We do this by limiting the domain.

For the sine function we select that portion whose ordered pairs are defined for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. See figure 4–4. Note that this new function is one to one, since there is no horizontal line that would intersect the graph in more than one point. Since it is one to one, it has an inverse function. This inverse function is denoted by $\sin^{-1}$, which is read "inverse sine function." The ordered pairs of this inverse sine function are the reversals of the ordered pairs of the one-to-one portion of the sine function we selected.

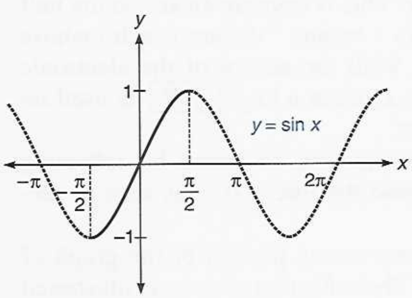


Figure 4–4

We know that if $y = \sin^{-1}x$, then the ordered pair (x,y) is in the inverse sine function. The ordered pair (y,x) is therefore in the selected one-to-one portion of the sine function. If (y,x) is in this part of the sine function, then $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$ and $x = \sin y$.

Note Remember that if an ordered pair is in the sine function, then the second element of the pair is the sine of the first.

Using the previous statements as a guide, we make the following definition of the inverse sine function.

The inverse sine function

$y = \sin^{-1}x$ means

1. $\sin y = x$
2. $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
3. $|x| \le 1$

Concept

If we think of $\sin^{-1}x$ as representing an angle, and since we know that angles between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ are in the first and fourth quadrants, we can interpret $y = \sin^{-1}x$ in the following way: "$\sin^{-1}x$ is the angle in quadrant I or quadrant IV whose sine is x." (We are referring to negative angles in quadrant IV.)

The domain of the inverse sine function is $-1 \le x \le 1$, or $|x| \le 1$, which is the range of the sine function. The range of the inverse sine function is

$-\frac{\pi}{2} \le \sin^{-1}x \le \frac{\pi}{2}$, which is the domain of the one-to-one portion of the sine function that we selected. Note that these are parts 2 and 3 of the definition. More explicitly,

$$\text{Domain}_{\sin^{-1}}\text{: } |x| \le 1$$

$$\text{Range}_{\sin^{-1}}\text{: } -\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

Another notation for $\sin^{-1}x$ is **arcsin** x. This is because an arc on the unit circle can represent an angle, and so arcsin x means "the arc (angle) whose sine is x, in quadrant I or quadrant IV." With the advent of the electronic calculator a third notation might be **invsin** x, since a key $\boxed{\text{INV}}$ is used on some calculators to find inverse sine values.

The graph of the inverse of any function can be found by reflecting each point in the graph of the function across the line $y = x$, as seen in section 4–1.

To graph $y = \sin^{-1}x$, we reflect the one-to-one portion of the graph of $y = \sin x$ (figure 4–4) across the line $y = x$. The reflecting process is illustrated in figure 4–5, and the final graph of $y = \sin^{-1}x$ is shown in figure 4–6.

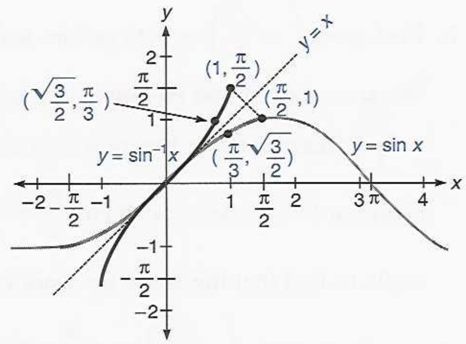
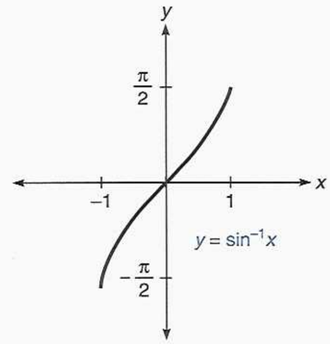


Figure 4–5



$Y_1=\sin^{-1} X$
RANGE $-3,3,.5,-2,2,.785$

Figure 4–6

Although the inverse trigonometric functions are defined using radian measure, we often want the result in degrees. For the inverse sine function we therefore want angles between $-90°$ and $90°$, since these correspond to $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ radians.

Example 4–2 A

1. Find $\sin^{-1}\frac{1}{2}$ in both radians and degrees.

To use the definition, we write $y = \sin^{-1}\frac{1}{2}$. Then we know that

$$\sin y = \tfrac{1}{2}$$

and

$$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$$

We know that $\sin\frac{\pi}{6} = \frac{1}{2}$, and $\frac{\pi}{6}$ is between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$, so $y = \frac{\pi}{6} = \sin^{-1}\frac{1}{2}$.

It is often easier to solve these problems if we restate them verbally. Remember that $\sin^{-1}\frac{1}{2}$ can be interpreted to mean "the angle in quadrant I or quadrant IV whose sine is $\frac{1}{2}$." We have memorized the fact that this is $\frac{\pi}{6}$.

Thus, $\sin^{-1}\frac{1}{2} = \frac{\pi}{6}$. Note that this corresponds to 30°.

2. Find $\arcsin\left(-\frac{\sqrt{2}}{2}\right)$ in both radians and degrees.

We are asked to find the angle in quadrant I or quadrant IV whose sine is $-\frac{\sqrt{2}}{2}$. Since the sine function is positive in quadrant I, we must look in quadrant IV. We know that $\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, so we use $\frac{\pi}{4}$ as a reference angle to find that the angle we want is $-\frac{\pi}{4}$.

Thus, $\arcsin\left(-\frac{\sqrt{2}}{2}\right) = -\frac{\pi}{4}$. This corresponds to $-45°$. ■

We memorized the sine values for certain angles, such as $\frac{\pi}{6}$ (30°), $\frac{\pi}{4}$ (45°), and $\frac{\pi}{3}$ (60°). In most cases, however, we can obtain only decimal approximations. For this we use calculators, as shown below and in sections 1–3, 1–4, 2–4, and 2–6. We will round radian answers to two decimal places and degree answers to one decimal place.

■ ***Example 4–2 B***

1. Find $\sin^{-1}0.5312$ in both radians and degrees.

Put the calculator in radian mode. Most calculators use either a $\boxed{\text{SIN}^{-1}}$ key or the $\boxed{\text{INV}}$ (or $\boxed{\text{ARC}}$ or $\boxed{\text{2nd}}$) before the $\boxed{\text{SIN}}$ key.

.5312 $\boxed{\text{INV}}$ $\boxed{\text{SIN}}$ or Display: $\boxed{0.560016290}$

.5312 $\boxed{\text{SIN}^{-1}}$

$\boxed{\text{TI-81}}$ $\boxed{\text{2nd}}$ $\boxed{\text{SIN}}$ .5312 $\boxed{\text{ENTER}}$

Thus, $\sin^{-1}0.5312 \approx 0.56$.

To obtain the result in degrees, put the calculator in degree mode before performing the calculations. When we round to one decimal place, we get 32.1°.

2. Find $\arcsin(-0.9249)$ in both radians and degrees.

We want the arc (angle) whose sine is -0.9249 and is in quadrant I or quadrant IV. The negative value tells us we want an angle in quadrant IV.

Make sure the calculator is in radian mode.

.9249 $\boxed{+/-}$ $\boxed{\text{INV}}$ $\boxed{\text{SIN}}$ (or $\boxed{\text{SIN}^{-1}}$) Display: $\boxed{-1.180772498}$

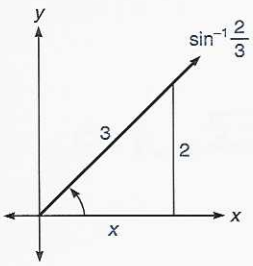
$\boxed{\text{TI-81}}$ $\boxed{\text{2nd}}$ $\boxed{\text{SIN}}$ $\boxed{(-)}$ .9249 $\boxed{\text{ENTER}}$

and round to two decimal places, or -1.18. Thus, $\arcsin(-0.9249) \approx -1.18$.

Performing the same calculator steps in degree mode gives $-67.7°$. ■

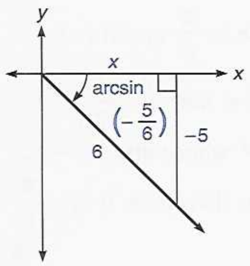
It is often important to be able to simplify expressions that involve combinations of the trigonometric and inverse trigonometric functions. This can often be done with the aid of the reference triangle we studied in section 2–4.

■ Example 4–2 C



1. Simplify $\tan(\sin^{-1}\frac{2}{3})$.

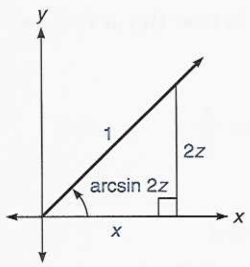
   Since $\frac{2}{3}$ is positive, the angle represented by $\sin^{-1}\frac{2}{3}$ is in quadrant I. (Remember, the range of the $\sin^{-1}$ function is quadrant I and quadrant IV.) Since $\sin^{-1}\frac{2}{3}$ means the angle in quadrant I whose sine is $\frac{2}{3}$, we can draw a reference triangle (section 2–4) in quadrant I for this angle. See the figure. Using the Pythagorean theorem, we calculate the third side of the triangle to be $\sqrt{5}$. From this we can see that the tangent of this angle is $\dfrac{2}{\sqrt{5}} = \dfrac{2}{\sqrt{5}} \cdot \dfrac{\sqrt{5}}{\sqrt{5}} = \dfrac{2\sqrt{5}}{5}$. Thus, $\tan(\sin^{-1}\frac{2}{3}) = \dfrac{2\sqrt{5}}{5}$.



2. Simplify $\sec[\arcsin(-\frac{5}{6})]$.

   $\text{Arcsin}(-\frac{5}{6})$ represents an angle in quadrant IV whose sine is $-\frac{5}{6}$. A reference triangle for such an angle is shown in the figure.

   We compute the length of side x to be $\sqrt{11}$. Now we can compute the cosine of this angle to be $\dfrac{\sqrt{11}}{6}$, and the secant of this angle is then the reciprocal of the cosine. That is, $\dfrac{6}{\sqrt{11}} = \dfrac{6}{\sqrt{11}} \cdot \dfrac{\sqrt{11}}{\sqrt{11}} = \dfrac{6\sqrt{11}}{11}$. Thus, $\sec[\arcsin(-\frac{5}{6})] = \dfrac{6\sqrt{11}}{11}$.



3. Simplify $\tan(\arcsin 2z)$, if $z > 0$.

   Since $z > 0$, then $2z$ is also positive, so $\arcsin 2z$ represents an angle in quadrant I whose sine is $2z$. A reference triangle for such an angle is shown in the figure.

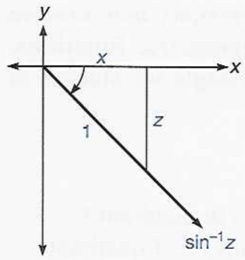
   We find the horizontal side x by the Pythagorean theorem.

$$1^2 = x^2 + (2z)^2$$
$$x^2 = 1 - 4z^2$$
$$x = \sqrt{1 - 4z^2}$$

   Since x is positive, we chose the positive solution when we took the square root.

   We can now find the tangent of this angle. We form the ratio of the side opposite the angle to the side adjacent to the angle to get $\dfrac{2z}{\sqrt{1 - 4z^2}}$. We will not rationalize this denominator.

   Thus, $\tan(\arcsin 2z) = \dfrac{2z}{\sqrt{1 - 4z^2}}$, if $z > 0$.



4. Compute $\cos(\sin^{-1}z)$, $z < 0$.

Since $z < 0$, $\sin^{-1}z$ represents an angle in quadrant IV. A reference triangle for such an angle is shown in the figure. Note that we do not write $-z$, but simply z, for the directed vertical distance, since z already represents a negative quantity.

We find the length x.

$$x^2 + z^2 = 1^2$$
$$x^2 = 1 - z^2$$
$$x = \sqrt{1 - z^2}$$

Now we compute the cosine of this angle. It is $\dfrac{\sqrt{1 - z^2}}{1}$ or $\sqrt{1 - z^2}$.

Thus, $\cos(\sin^{-1}z) = \sqrt{1 - z^2}$ if $z < 0$.

5. Compute $\sin^{-1}\left(\sin \dfrac{7\pi}{6}\right)$.

We know that $\sin \dfrac{7\pi}{6} = -\dfrac{1}{2}$ since its reference angle is $\dfrac{\pi}{6}$ and it is in quadrant III. (See section 2–3.) Thus, $\sin^{-1}\left(\sin \dfrac{7\pi}{6}\right) = \sin^{-1}\left(-\dfrac{1}{2}\right)$. Thus, we need the angle in quadrant I or quadrant IV whose sine is $-\frac{1}{2}$. The angle is in quadrant IV since its sine is negative; therefore, it is $-\dfrac{\pi}{6}$ in radians or $-30°$.

Part 5 of example 4–2 C shows that $\sin^{-1}(\sin x)$ is not necessarily x. If, however, x is an angle between $-\dfrac{\pi}{2}$ and $\dfrac{\pi}{2}$, then this is true (try a few examples). Thus, we note:

$$\sin^{-1}(\sin x) = x \text{ if and only if } -\frac{\pi}{2} \le x \le \frac{\pi}{2}$$

Also,

$$\sin(\sin^{-1}x) = x \text{ if and only if } -1 \le x \le 1$$

This last restriction applies simply because the domain of the inverse sine function is $-1 \le x \le 1$.

Mastery points

Can you

- Compute exact and approximate values for the inverse of the sine function?
- Simplify expressions that combine the trigonometric functions and the inverse sine function?
- State the domain and range of the inverse sine function?

Exercise 4–2

1. Sketch the graph of the inverse sine function.
2. State the domain and range of the inverse sine function.

Find exact values for each of the following expressions. State the results in both radians and degrees.

3. $\sin^{-1}(-\frac{1}{2})$
4. $\arcsin\frac{\sqrt{3}}{2}$
5. $\sin^{-1}0$
6. $\arcsin\left(-\frac{\sqrt{2}}{2}\right)$
7. $\arcsin\left(-\frac{\sqrt{3}}{2}\right)$
8. $\sin^{-1}1$

Find approximate values for the following expressions in both radians and degrees. Round the values for radians to two decimal places and for degrees to one decimal place.

9. $\sin^{-1}0.8823$
10. $\arcsin 0.8253$
11. $\sin^{-1}0.9323$
12. $\sin^{-1}0.6442$
13. $\sin^{-1}(-0.9976)$
14. $\sin^{-1}(-0.2955)$
15. $\arcsin(-0.2571)$
16. $\arcsin(-0.9888)$

Simplify each of the following expressions.

17. $\tan(\arcsin\frac{5}{8})$
18. $\cos(\arcsin\frac{3}{5})$
19. $\sec[\sin^{-1}(-\frac{2}{3})]$
20. $\tan[\arcsin(-0.8)]$
21. $\cot\left(\sin^{-1}\frac{\sqrt{3}}{5}\right)$
22. $\csc\left(\sin^{-1}\frac{\sqrt{2}}{6}\right)$
23. $\cos(\arcsin 0.3)$
24. $\tan(\arcsin 0.4)$
25. $\cos(\sin^{-1}z)$, $z > 0$
26. $\cos(\sin^{-1}3z)$, $z < 0$
27. $\tan[\sin^{-1}(1+z)]$, $1+z<0$
28. $\cos(\arcsin\sqrt{z})$
29. $\sec(\arcsin\sqrt{2z})$
30. $\cot(\sin^{-1}\sqrt{z-1})$
31. $\sin^{-1}\left(\sin\frac{\pi}{6}\right)$
32. $\arcsin\left(\tan\frac{\pi}{4}\right)$
33. $\arcsin\left(\cos\frac{2\pi}{3}\right)$
34. $\sin^{-1}\left(\sin\frac{11\pi}{6}\right)$
35. $\sin^{-1}(\cos 0)$
36. $\arcsin\left(\sin\frac{5\pi}{6}\right)$

37. Recall that a function f is periodic if there is a number p, $p>0$, such that $f(x)=f(x+p)$ for all x in the domain of the function. Can a periodic function be one to one? Give a reason for your answer.

4–3 The inverse cosine and inverse tangent functions

The inverse of the cosine function is formed in the same way as the inverse of the sine function. We first select a one-to-one portion. This is chosen to be between 0 and π. See figure 4–7.

The graph of $y = \cos^{-1}x$ is the reflection of this portion of the cosine function across the line $y = x$ (shown in figure 4–7 by a dashed line). The graph of the inverse cosine function is shown separately in figure 4–8.

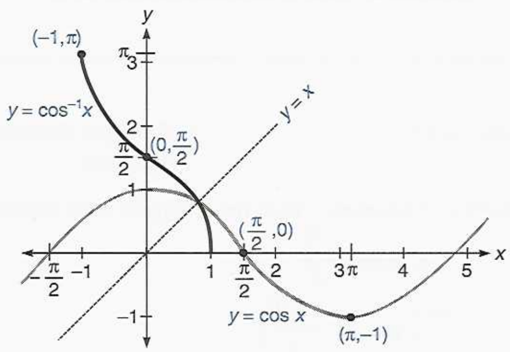
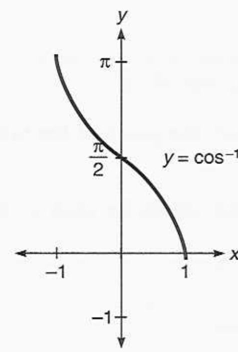


Figure 4–7



$Y_1 = \cos^{-1} X$

RANGE −3,3,.5,−.5,3.5,.785

Figure 4–8

We can see from the graph that

$$\text{Domain}_{\cos^{-1}}: |x| \leq 1$$
$$\text{Range}_{\cos^{-1}}: 0 \leq y \leq \pi$$

The inverse cosine function is defined as follows:

The inverse cosine function

$y = \cos^{-1}x$ means

1. $\cos y = x$
2. $0 \leq y \leq \pi$
3. $|x| \leq 1$

Concept

$\cos^{-1}x$ is the angle in quadrant I or quadrant II whose cosine is x.

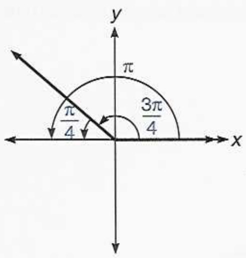
Arccos x and **invcos** x mean the same thing as $\cos^{-1}x$. Note that in degrees the inverse cosine varies from 0° to 180°.

■ ***Example 4–3 A***

1. Find $\cos^{-1}\frac{1}{2}$ in both radians and degrees.

We want the angle in quadrant I or quadrant II whose cosine is $\frac{1}{2}$. Since this is a positive value, the angle is in quadrant I. We know $\cos\frac{\pi}{3} = \frac{1}{2}$, so $\cos^{-1}\frac{1}{2} = \frac{\pi}{3}$. Note that this is equivalent to 60°.

2. Find $\arccos\left(-\frac{\sqrt{2}}{2}\right)$ in both radians and degrees.



We want the angle in quadrant I or quadrant II whose cosine is $-\frac{\sqrt{2}}{2}$.

Since this is a negative value, the angle is in quadrant II. We know $\cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$, so we use $\frac{\pi}{4}$ as a reference angle in quadrant II. This gives us $\pi - \frac{\pi}{4} = \frac{3\pi}{4}$ for our angle. See the figure.

Thus, $\arccos\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$, which is equivalent to 135°. ■

As with the inverse sine function, we sometimes need approximate answers. This is illustrated in example 4–3 B.

■ ***Example 4–3 B***

1. Find $\cos^{-1}0.9638$ in both radians and degrees.

We are looking for the angle in quadrant I or quadrant II whose cosine is 0.9638. Since this is a positive value, the angle is in quadrant I. Putting the calculator in radian mode,

.9638 [INV] [COS] (or [COS^{-1}]) Display: [0.269890866]

[TI-81] [2nd] [COS] .9638 [ENTER]

Thus, $\cos^{-1}0.9638 \approx 0.27$.

If the calculation is done with the calculator in degree mode the result is approximately 15.5°.

2. Find $\arccos(-0.5141)$ in both radians and degrees.

We are looking for the angle in quadrant I or quadrant II whose cosine is -0.5141. Since this value is negative, the angle is in quadrant II.

In radian mode,

.5141 [+/−] [INV] [COS] Display: [2.110754365]

[TI-81] [2nd] [COS] [(−)] .5141 [ENTER]

Thus, $\arccos(-0.5141) \approx 2.11$. When calculated in degree mode the result is approximately 120.9°. ■

The inverse of the tangent function is formed in the same way as the inverses of the sine and cosine functions. We first select a one-to-one portion. This is chosen to be between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. (In fact, this is the same as our basic tangent cycle from section 3–1.) The graph of $y = \tan^{-1}x$ is the reflection of this basic cycle across the line $y = x$. The graph of the basic cycle and its reflection are shown in figure 4–9 and the graph of the inverse tangent function is shown in figure 4–10.

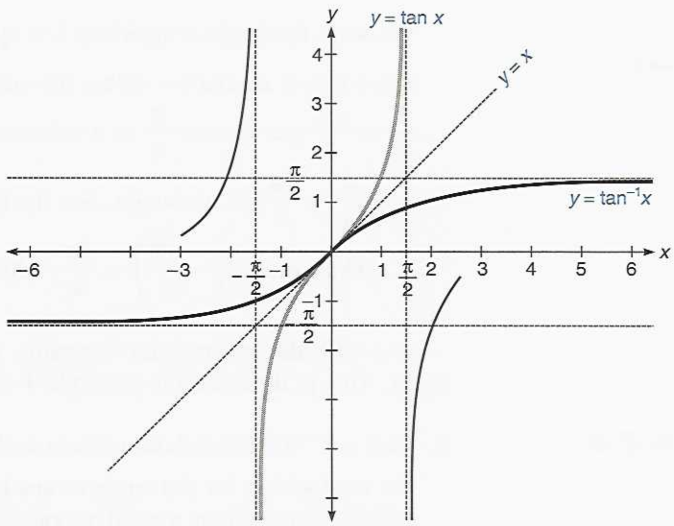
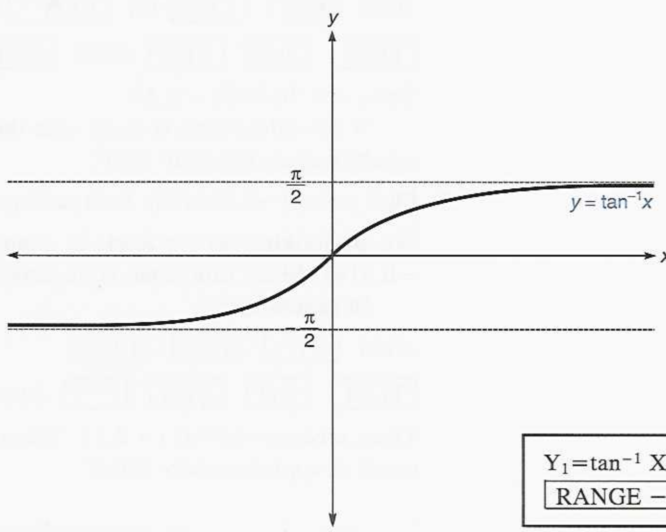


Figure 4–9



$Y_1=\tan^{-1} X$
RANGE −3,3,.5,−2,2,.785

Figure 4–10

The domain of the inverse tangent function is all the reals, and the range is all values between $-\frac{\pi}{2}$ and $\frac{\pi}{2}$. Observe that the range is in quadrants I and IV, as is the range of the inverse sine function, except that the range of the inverse tangent function does not include the points $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ themselves.

Thus,

$$\text{Domain}_{\tan^{-1}}: R$$

$$\text{Range}_{\tan^{-1}}: -\frac{\pi}{2} < y < \frac{\pi}{2}$$

The inverse tangent function

$y = \tan^{-1}x$ means

1. $\tan y = x$ and
2. $-\frac{\pi}{2} < y < \frac{\pi}{2}$

Concept

$\tan^{-1}x$ means the angle in quadrant I or quadrant IV whose tangent is x.

Note Since x can take on any value, we do not restrict it in the definition as we did for the inverse sine and inverse cosine functions.

Arctan x and **invtan** x are other notations for $\tan^{-1}x$.

■ ***Example 4–3 C***

1. Find $\tan^{-1}\sqrt{3}$ in both radians and degrees.

We want the angle in quadrant I or quadrant IV whose tangent is $\sqrt{3}$. Since this is a positive value, the angle is in quadrant I. We know $\tan \frac{\pi}{3} = \sqrt{3}$, so $\tan^{-1}\sqrt{3} = \frac{\pi}{3}$. This is 60°, also.

2. Find arctan(−1) in both radians and degrees.

We want the angle in quadrant I or quadrant IV whose tangent is −1. Since this is a negative value, the angle is in quadrant IV. We know $\tan \frac{\pi}{4} = 1$, so we use $\frac{\pi}{4}$ as a reference angle in quadrant IV. This gives us $-\frac{\pi}{4}$, or −45°, for our angle.

Thus, $\arctan(-1) = -\frac{\pi}{4}$, or $-45°$. ■

Example 4–3 D illustrates obtaining approximate values with the calculator.

Example 4–3 D

Find arctan(−1.9208) in both radians and degrees.

We are looking for the angle in quadrant I or quadrant IV whose tangent is −1.9208. Since this value is negative the angle is in quadrant IV.

1.9208 [+/−] [INV] [TAN] (or [TAN^{-1}]) Display: [−1.090791948]

[TI-81] [2nd] [TAN] [(−)] 1.9208 [ENTER]

Thus, the result is about −1.09 radians and −62.5°.

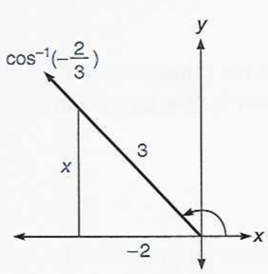
The domains and ranges of the inverses of the sine, cosine, and tangent functions are summarized in table 4–1. Also indicated are the quadrants to which the ranges correspond.

Function	Domain	Range	Quadrants
$y = \sin^{-1}x$	$\lvert x\rvert \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$	I, IV
$y = \cos^{-1}x$	$\lvert x\rvert \le 1$	$0 \le y \le \pi$	I, II
$y = \tan^{-1}x$	R	$-\frac{\pi}{2} < y < \frac{\pi}{2}$	I, IV

Table 4–1

We can simplify expressions that involve combinations of the trigonometric and inverse trigonometric functions using the reference triangle, as we did in the previous section.

Example 4–3 E

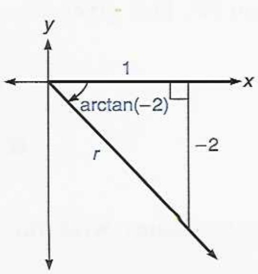


1. Simplify $\tan[\cos^{-1}(-\frac{2}{3})]$.

   Since $-\frac{2}{3}$ is negative, the angle represented by $\cos^{-1}(-\frac{2}{3})$ is in quadrant II. (Remember, the range of this function is quadrants I and II.) Since $\cos^{-1}(-\frac{2}{3})$ means the angle in quadrant II whose cosine is $-\frac{2}{3}$, we draw a reference triangle in quadrant II for this angle. See the figure.

   Using the Pythagorean theorem, we calculate the third side of the triangle to be $\sqrt{5}$. From this we can see that the tangent of this angle is $\frac{\sqrt{5}}{-2} = -\frac{\sqrt{5}}{2}$.

   Thus, $\tan\left[\cos^{-1}\left(-\frac{2}{3}\right)\right] = -\frac{\sqrt{5}}{2}$.

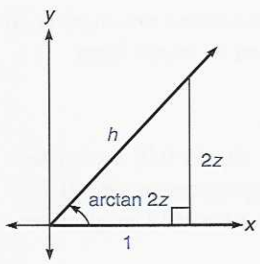


2. Simplify sec[arctan(−2)].

   Arctan(−2) represents an angle in quadrant IV whose tangent is −2. A reference triangle for such an angle is shown in the figure.

   Using the Pythagorean theorem, we compute the length of the hypotenuse to be $\sqrt{5}$. From this triangle we can find the secant of this angle. We see that the cosine is $\frac{1}{\sqrt{5}}$, so the secant is the reciprocal, or $\sqrt{5}$.

   Thus, $\sec[\arctan(-2)] = \sqrt{5}$.



3. Simplify $\sin(\arctan 2z)$, if $z > 0$.

Since $z > 0$, then $2z$ is also positive, so $\arctan 2z$ represents an angle in quadrant I whose tangent is $2z$. A reference triangle for such an angle is shown in the figure.

We find the hypotenuse h using the Pythagorean theorem.

$$1^2 + (2z)^2 = h^2$$
$$1 + 4z^2 = h^2$$
$$\sqrt{1 + 4z^2} = h$$

We can now find the sine of this angle. We form the ratio of the side opposite the angle to the hypotenuse to get $\frac{2z}{\sqrt{1 + 4z^2}}$. We will not rationalize this denominator.

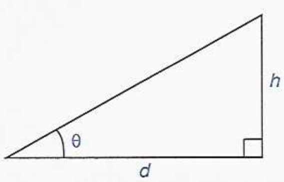
Thus, $\sin(\arctan 2z) = \frac{2z}{\sqrt{1 + 4z^2}}$ if $z > 0$.

4. Find $\tan^{-1}\left[\sin\left(-\frac{\pi}{2}\right)\right]$.

$$\tan^{-1}\left[\sin\left(-\frac{\pi}{2}\right)\right] = \tan^{-1}(-1)$$
$$= -\frac{\pi}{4} \text{ or } -45°$$

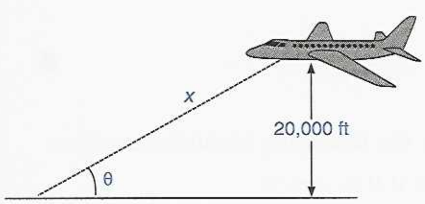
One application of the inverse trigonometric functions is to describe an angle in a given situation with a mathematical expression.

■ Example 4–3 F



1. Describe angle θ in the figure using an inverse trigonometric function.

We can see that the tangent of θ is $\frac{h}{d}$, so "θ is an angle whose tangent is $\frac{h}{d}$." This is what $\tan^{-1}\frac{h}{d}$ means, so $\theta = \tan^{-1}\frac{h}{d}$.



2. A jet aircraft is flying at 20,000 feet. If x represents the distance from a ground observer to the aircraft (also in feet), describe the angle of elevation from the observer to the aircraft in terms of an inverse trigonometric function.

If we represent the situation as shown in the figure, where θ indicates the angle of elevation, we see that θ is an angle whose sine is $\frac{20{,}000}{x}$. This is what the expression $\sin^{-1}\frac{20{,}000}{x}$ means.

Thus, since $\sin\theta = \frac{20{,}000}{x}$, $\theta = \sin^{-1}\frac{20{,}000}{x}$.

Another way in which the inverse trigonometric functions are used is to describe one of the solutions to a trigonometric equation in exact form.

■ *Example 4–3 G*

1. $\sin\theta = 0.7$. Describe one value of θ in exact form.

   Since the equation tells us that θ is an angle whose sine is 0.7, we write $\theta = \sin^{-1}0.7$. This expression is exact. We could approximate $\sin^{-1}0.7$ with a calculator, but the decimal number we obtained would only be an approximation.

   Thus, if $\sin\theta = 0.7$, one exact value of θ is $\sin^{-1}0.7$.

2. $\tan\theta = z$. Describe one value of θ in exact form.

   We see that $\theta = \tan^{-1}z$, so one value would be $\tan^{-1}z$.

3. $\cos 2\theta = 0.55$. Describe one value of θ in exact form.

   $2\theta = \cos^{-1}0.55$, so $\theta = \dfrac{\cos^{-1}0.55}{2}$ is one value of θ.

4. $3\sin\theta = 0.69$. Describe one value of θ in exact form.

   If we divide both sides by 3, we see that $\sin\theta = 0.23$. Thus, $\theta = \sin^{-1}0.23$ gives one exact value of θ.

5. $A\sin Bx = C$. Describe one value of x in exact form.

   Dividing both sides by A:

   $$\sin Bx = \frac{C}{A}$$

   Then,

   $$Bx = \sin^{-1}\frac{C}{A}$$

   Dividing both sides by B:

   $$x = \frac{\sin^{-1}\frac{C}{A}}{B}$$

   Thus, one value of x that satisfies the equation $A\sin Bx = C$ is $\dfrac{\sin^{-1}\frac{C}{A}}{B}$. ■

Finally, it is not too hard to verify that the following identities are true:

$$\cos^{-1}(\cos x) = x \text{ if and only if } 0 \le x \le \pi$$
$$\cos(\cos^{-1}x) = x \text{ if and only if } -1 \le x \le 1$$
$$\tan^{-1}(\tan x) = x \text{ if and only if } -\frac{\pi}{2} < x < \frac{\pi}{2}$$
$$\tan(\tan^{-1}x) = x \text{ for all } x$$

Mastery points

Can you

- Compute exact and approximate values for the inverses of the cosine and tangent functions?
- Simplify expressions that combine the trigonometric functions and their inverses?
- State the domains and ranges of these inverse trigonometric functions?
- Apply these inverse trigonometric functions to describe angles in physical situations?
- Apply these inverse trigonometric functions to give one exact solution to trigonometric equations?

Exercise 4–3

1. Sketch the graph of each function.
 a. inverse sine
 b. inverse cosine
 c. inverse tangent

2. State the domain and range for each function.
 a. inverse sine
 b. inverse cosine
 c. inverse tangent

Find exact values for each of the following expressions in both radians and degrees.

3. $\cos^{-1}(-\frac{1}{2})$ **4.** $\arccos \frac{\sqrt{3}}{2}$ **5.** $\cos^{-1}0$ **6.** $\arccos\left(-\frac{\sqrt{2}}{2}\right)$ **7.** $\arcsin \frac{\sqrt{3}}{2}$

8. $\tan^{-1}1$ **9.** $\arctan(-\sqrt{3})$ **10.** $\cos^{-1}\frac{\sqrt{2}}{2}$ **11.** $\tan^{-1}\left(-\frac{\sqrt{3}}{3}\right)$ **12.** $\cos^{-1}\left(-\frac{\sqrt{3}}{2}\right)$

Find approximate values for the following expressions, in both degrees and radians. Round degrees to tenths and radians to hundredths.

13. $\sin^{-1}0.8823$ **14.** arctan 11.08 **15.** arccos 0.8253 **16.** arcsin 0.6131
17. $\tan^{-1}0.9316$ **18.** $\cos^{-1}0.6442$ **19.** arcsin 0.7961 **20.** $\sin^{-1}0.8776$
21. $\sin^{-1}(-0.9976)$ **22.** $\cos^{-1}(-0.2955)$ **23.** $\arctan(-0.2553)$ **24.** $\arccos(-0.9888)$
25. $\arccos(-0.9902)$ **26.** $\tan^{-1}(-3.4776)$

Simplify each of the following expressions.

27. $\cos(\tan^{-1}\frac{3}{5})$ **28.** $\sin(\cos^{-1}\frac{1}{4})$ **29.** $\sin(\arccos \frac{5}{8})$ **30.** $\cos(\arctan 2)$

31. $\tan[\cos^{-1}(-\frac{2}{3})]$ **32.** $\sin[\arccos(-0.8)]$ **33.** $\sin(\tan^{-1}\sqrt{5})$ **34.** $\csc\left(\cos^{-1}\frac{\sqrt{2}}{6}\right)$

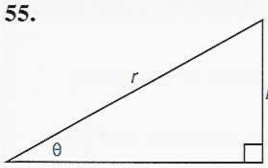
35. $\cos(\arctan 0.3)$ **36.** $\sin(\arctan 0.4)$ **37.** $\sin(\cos^{-1}z)$, $z > 0$ **38.** $\cos(\arccos z)$, $z > 0$
39. $\tan(\arccos z)$, $z > 0$ **40.** $\sin(\tan^{-1}z)$, $z > 0$ **41.** $\cos(\arctan z)$, $z < 0$ **42.** $\tan(\cos^{-1}z)$, $z < 0$
43. $\sin(\cos^{-1}3z)$, $z < 0$ **44.** $\cos(\arctan 2z)$, $z > 0$ **45.** $\sec[\tan^{-1}(1 + z)]$, $z > 0$ **46.** $\sin(\arccos \sqrt{z})$

47. $\cos(\arctan \sqrt{2z})$ **48.** $\tan(\cos^{-1}\sqrt{z - 1})$ **49.** $\tan^{-1}\left(\sin \frac{\pi}{2}\right)$ **50.** $\cos^{-1}\left(\sin \frac{7\pi}{6}\right)$

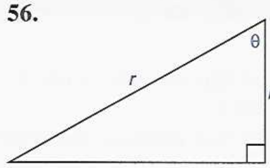
51. $\cos^{-1}\left(\cos \frac{11\pi}{6}\right)$ **52.** $\tan^{-1}(\cos 0)$ **53.** $\arccos\left(\tan \frac{5\pi}{4}\right)$ **54.** $\cos^{-1}\left(\cos \frac{3\pi}{2}\right)$

In the following problems state the angle θ shown in each diagram in terms of an inverse trigonometric function.

55.



56.



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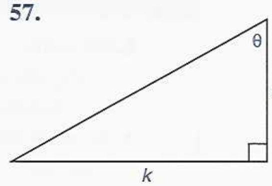
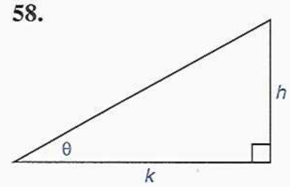


58.



59. A picture hangs on a wall so that the bottom of the picture is 5 feet above the floor. The picture is 2 feet high. Describe the angle subtended by the picture at the eye of an observer, in terms of an inverse trigonometric function, if the observer's eye is also 5 feet above the floor and the observer is x feet away from the wall.

60. A radar antenna will track the launch of a rocket from a point 12,500 feet from the rocket. Both are at the same ground elevation at launch. If a represents the altitude of the rocket in feet, describe the angle of elevation of the rocket at the radar site in terms of an inverse trigonometric function.

61. An aircraft is flying toward an airport at an elevation of 3,500 feet above the airport. Describe the angle of depression of the airport at the aircraft in terms of the distance z from the aircraft directly to the airport, using an inverse trigonometric function.

In the following problems describe one value of θ, or x, in exact form, in terms of an inverse trigonometric function.

62. $\sin\theta = 0.75$

63. $\cos\theta = -0.8$

64. $\tan\theta = 3$

65. $\tan\theta = 4.1$

66. $2\sin\theta = 1.6$

67. $3\tan\theta = 5$

68. $\frac{1}{2}\sin\theta = -0.1$

69. $\dfrac{\tan\theta}{5} = 10$

70. $\sin 2\theta = 0.76$

71. $\tan 3\theta = 9$

72. $\cos\dfrac{\theta}{3} = -0.42$

73. $\sin\dfrac{3\theta}{2} = -0.56$

74. $4\cos 3\theta = 3$

75. $2\sin 4\theta = 1.5$

76. $\dfrac{3\cos 2\theta}{8} = \dfrac{7}{40}$

77. $\dfrac{6\sin 5\theta}{5} = \dfrac{10}{13}$

78. $\dfrac{A\cos Bx}{C} = D$

79. $A\tan(Bx + C) = D$

80. $\cos(x - 2) = 0.2$

81. $\sin(2x + 3) = 0.6$

82. Many computer languages only provide an arctangent function. In these situations we must program our own arcsine and arccosine functions. Use appropriate reference triangles to show that the following are identities.

a. $$\arcsin(x) = \begin{cases} -\dfrac{\pi}{2} & \text{if } x = -1 \\ \arctan\left(\dfrac{x}{\sqrt{1-x^2}}\right) & \text{if } |x| < 1 \\ \dfrac{\pi}{2} & \text{if } x = 1 \end{cases}$$

b. $$\arccos(x) = \begin{cases} \arctan\left(\dfrac{\sqrt{1-x^2}}{x}\right) & \text{if } 0 < x \le 1 \\ \dfrac{\pi}{2} & \text{if } x = 0 \\ \arctan\left(\dfrac{\sqrt{1-x^2}}{x}\right) + \pi & \text{if } -1 \le x < 0 \end{cases}$$

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** Undergraduate students may choose to defer repayment until six months after graduation or ceasing to be enrolled at least half time in school. Interest only and immediate repayment options also available.

*** A 0.25% interest rate reduction is available for borrowers who elect to have monthly principal and interest payments transferred electronically from a savings or checking account. The interest rate reduction will begin when automatic principal and interest payments start, and will remain in effect as long as automatic payments continue without interruption. This reduced interest rate will return to contract rate if automatic payments are cancelled, rejected or returned for any reason. Upon request, borrowers are also entitled to an additional 0.25% interest rate reduction if (1) the first 36 payments of principal and interest are paid on time, and (2) at any time prior to the 36th on time payment, the borrower who receives the monthly bill elects to have monthly principal and interest payments transferred electronically from a savings or checking account, and continues to make such automatic payments through the 36th payment. This reduced interest rate will not be returned to contract rate if, after receiving the benefit, the borrower discontinues automatic electronic payment. The lender and servicer reserve the right to modify or discontinue borrower benefit programs (other than the co-signer release benefit) at any time without notice.

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4–4 The inverse cotangent, secant, and cosecant functions

The reciprocal trigonometric functions (cotangent, secant, cosecant) and their inverses were useful for computations before the advent of electronic calculating devices like calculators and computers; today they have no practical use in computation. This is why calculators do not have keys for the reciprocal functions, and why programming languages, such as FORTRAN, BASIC, or Pascal, do not support these functions. These functions still have value in expressing certain expressions in higher mathematics, however, and that is why we study them here.

The inverse cotangent function

We define the inverse cotangent function by reversing the ordered pairs in the basic cotangent cycle; that is, we restrict the domain to $0 < x < \pi$. Figure 4–11 shows the inverse cotangent function, $\cot^{-1}$. The domain is R, and the range is $0 < y < \pi$. As we might suspect **arccot** x also means $\cot^{-1}x$.

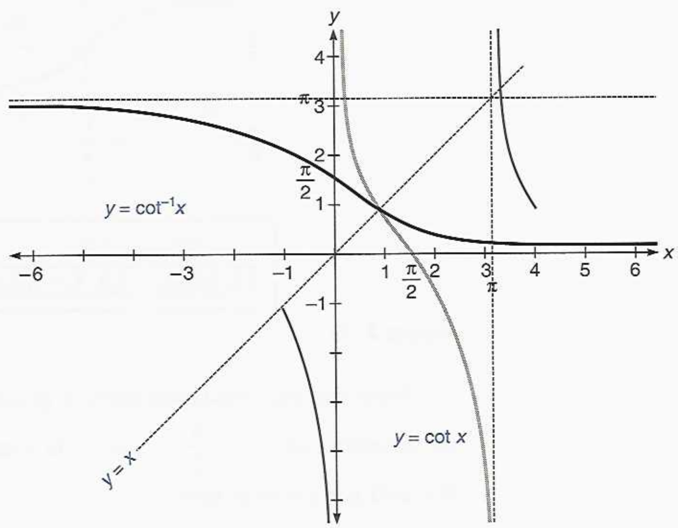


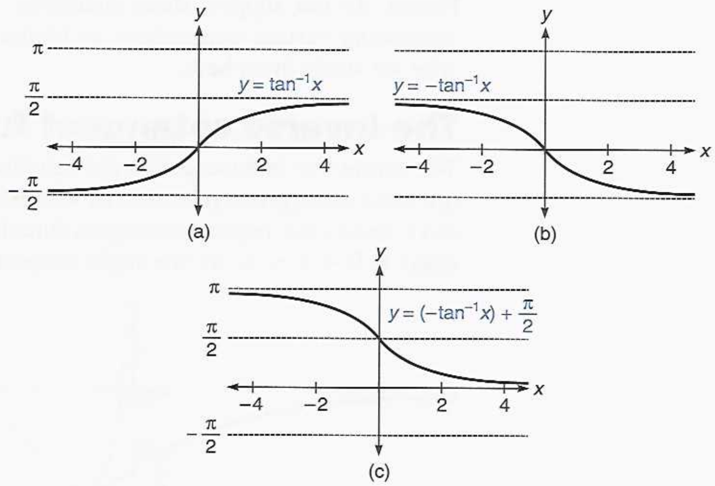
Figure 4–11

> **The inverse cotangent function**
>
> $y = \cot^{-1}x$ means
>
> 1. $\cot y = x$
> 2. $0 < y < \pi$

There are several ways to compute values of the inverse cotangent function. One way is to use the identity

$$\cot^{-1}x = \frac{\pi}{2} - \tan^{-1}x$$

This identity can be seen by considering the graphs of $y = \cot^{-1}x$ (figure 4–11) and $y = \tan^{-1}x$ (figure 4–12). Figure 4–12 shows the sequence of operations that will transform one graph into the other. Part (a) shows $y = \tan^{-1}x$. Part (b) shows the transformation caused by the scaling factor -1. Part (c) shows the vertical shift caused by adding $\frac{\pi}{2}$. This result is the same as the graph of the inverse cotangent function.



Y1=π/2−tan⁻¹ X
RANGE −3,3,.5,−.5,3.5,.785

Figure 4–12

Note that this transformation of graphs does not absolutely guarantee that the identity $\cot^{-1}x = \frac{\pi}{2} - \tan^{-1}x$ is true. It should be proven algebraically. We will not prove it here.

The inverse secant function

Recall the identity $\sec \theta = \frac{1}{\cos \theta}$. We use this to define the inverse of the secant function. Suppose we stated that $y = \sec^{-1}x$. Then $\sec y = x$. We proceed as shown.

$y = \sec^{-1}x$	An expression we want to define
$\sec y = x$	We expect the expression to have this property
$\dfrac{1}{\cos y} = x$	$\sec\theta = \dfrac{1}{\cos\theta}$
$\cos y = \dfrac{1}{x}$	$\dfrac{1}{\cos y} = x$; $1 = x\cos y$; $\dfrac{1}{x} = \cos y$
$y = \cos^{-1}\dfrac{1}{x}$	Since y is the angle whose cosine is $\dfrac{1}{x}$
$\sec^{-1}x = \cos^{-1}\dfrac{1}{x}$	

We use this sequence of steps to motivate our definition. As expected, **arcsec** x also means $\sec^{-1}x$.

The inverse secant function

$$\sec^{-1}x = \cos^{-1}\frac{1}{x} \text{ if } |x| \geq 1$$

We require $|x| \geq 1$ so that $\left|\dfrac{1}{x}\right| \leq 1$, as required by the inverse cosine function. The range of the secant function is the range of the inverse cosine function since the latter defines the former, except that $\dfrac{\pi}{2}$ is not in the range. This is because $\dfrac{\pi}{2} = \cos^{-1}0$, and there is no value of x such that $\dfrac{1}{x} = 0$.

Inverse cosecant function

The identity $\csc\theta = \dfrac{1}{\sin\theta}$, and reasoning similar to that above, leads to the following definition for the inverse cosecant (**arcsecant**) function.

The inverse cosecant function

$$\csc^{-1}x = \sin^{-1}\frac{1}{x} \text{ if } |x| \geq 1$$

For reasons similar to those stated earlier, the domain of the inverse cosecant function is $|x| \geq 1$, and the range is the same as that of the inverse sine function, except for 0, which is $\sin^{-1}0$, and $\dfrac{1}{x}$ can not take on the value 0.

Note It is not worth the effort to study the graphs of the functions of this section. We presented the graph of the inverse cotangent function only to justify the identity presented earlier.

Summary of properties

Table 4–2 summarizes the domains and ranges of the functions introduced previously.

Function	Domain	Range	Quadrants
$y = \cot^{-1}x$	R	$0 < y < \pi$	I, II
$y = \sec^{-1}x$	$\lvert x \rvert \geq 1$	$0 \leq y \leq \pi, y \neq \frac{\pi}{2}$	I, II
$y = \csc^{-1}x$	$\lvert x \rvert \geq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, y \neq 0$	I, IV

Table 4–2

Note in this table that quadrant I always corresponds to a nonnegative domain element (a nonnegative value of x), and quadrant II or IV corresponds to negative domain elements.

■ Example 4–4 A

Find the given values in both radians and degrees. Round radians to two decimal places and degrees to one decimal place where necessary.

1. $\cot^{-1}(-4)$

$= \frac{\pi}{2} - \tan^{-1}(-4)$ $\quad$ $\cot^{-1}x = \frac{\pi}{2} - \tan^{-1}x$

≈ 2.90 (radians) $\quad$ Calculator in radian mode

$= 90° - \tan^{-1}(-4)$

$\approx 166.0°$ $\quad$ Calculator in degree mode

Thus, $\cot^{-1}(-4) \approx 166.0°$ or 2.90 (radians).

2. $\sec^{-1} 2$

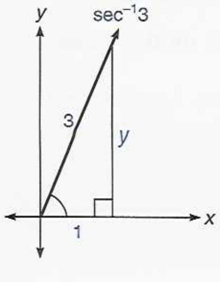
$= \cos^{-1}\frac{1}{2}$ $\quad$ Definition of $\sec^{-1}\theta$

$= 60°$ or $\frac{\pi}{3}$

Thus, $\sec^{-1}2 = 60°$ or $\frac{\pi}{3}$. ■

As we saw in section 4–3 some expressions that involve both the trigonometric and inverse trigonometric functions can be simplified by using a reference triangle.

■ Example 4–4 B



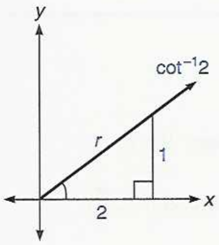
Simplify the expression.

1. $\sin(\sec^{-1}3)$

$\sec^{-1}3 = \cos^{-1}\frac{1}{3}$, by definition. This is a first quadrant angle. The figure shows a reference triangle in this quadrant in which the cosine of the angle is $\frac{1}{3}$.

$y = \sqrt{3^2 - 1^2} = \sqrt{8} = 2\sqrt{2}$

The sine of this angle is $\frac{\text{opposite}}{\text{hypotenuse}} = \frac{2\sqrt{2}}{3}$.

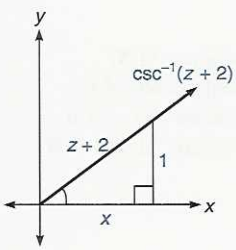


2. $\sec(\cot^{-1}2)$
$\cot^{-1}2$ is an angle in quadrant I. Since its cotangent is 2, its tangent is $\frac{1}{2}$. A reference triangle for a quadrant I angle with tangent $\frac{1}{2}$ is shown in the figure.

$r = \sqrt{5}$ (Pythagorean theorem), so the cosine of the angle is $\dfrac{2}{\sqrt{5}}$,

and therefore the secant is $\dfrac{\sqrt{5}}{2}$. Thus, $\sec(\cot^{-1}2) = \dfrac{\sqrt{5}}{2}$.

Note The identity $\cot^{-1}x = \dfrac{\pi}{2} - \tan^{-1}x$ would not be helpful in simplifying this expression. It is used for computing values of $\cot^{-1}x$.



3. $\cos(\csc^{-1}(z + 2)),\ z > 0$
$\csc^{-1}(z + 2) = \sin^{-1}\dfrac{1}{z + 2}$ by definition. Since $\dfrac{1}{z + 2}$ is positive,

$\sin^{-1}\dfrac{1}{z + 2}$ is an angle in quadrant I (see the figure). We find y by the Pythagorean theorem.

$$\begin{aligned}(z + 2)^2 &= 1^2 + y^2\\ z^2 + 4z + 4 - 1 &= y^2\\ \sqrt{z^2 + 4z + 3} &= y\end{aligned}$$

The cosine of the angle is $\dfrac{\text{adjacent}}{\text{hypotenuse}} = \dfrac{y}{z + 2} = \dfrac{\sqrt{z^2 + 4z + 3}}{z + 2}$. ■

Mastery points

Can you

- State the domain and range of the inverse cotangent, cosecant, and secant functions?
- Find exact values, in both radians and degrees, for expressions of the form $\cot^{-1}x$, $\csc^{-1}x$, and $\sec^{-1}x$, for appropriate values of x, using the definitions of these functions?
- Find approximate values, in both radians and degrees, for expressions of the form $\sin^{-1}x$, $\cos^{-1}x$, and $\tan^{-1}x$, using a calculator and the definitions of these functions?
- Simplify expressions that involve combinations of the trigonometric and inverse trigonometric functions, using a reference triangle where appropriate?

Exercise 4–4

Compute the exact value of the following expressions.

1. $\csc^{-1}2$ **2.** $\operatorname{arcsec}\left(-\frac{2\sqrt{3}}{3}\right)$ **3.** arccot 1 **4.** arccsc(−1) **5.** $\sec^{-1}(-2)$

6. $\cot^{-1}(-\sqrt{3})$ **7.** $\operatorname{arccsc}\frac{2\sqrt{3}}{3}$ **8.** $\csc^{-1}\sqrt{2}$ **9.** arccot 0 **10.** $\operatorname{arcsec}(-\sqrt{2})$

Compute approximate values of the following expressions in both radians (to hundredths) and degrees (to tenths).

11. $\csc^{-1}3.3534$ **12.** $\cot^{-1}0.5080$ **13.** arcsec(−2.9986) **14.** arccsc(−2.5087)
15. arccot 5.1997 **16.** $\sec^{-1}(-2.0126)$ **17.** $\sec^{-1}(-11.1261)$ **18.** $\csc^{-1}(-3.8898)$
19. arccsc 3.1790 **20.** arccot(−8.3534)

Simplify the following expressions.

21. $\sin(\csc^{-1}3)$ **22.** $\cos(\cot^{-1}2)$ **23.** cot(arcsec 4) **24.** $\sin(\sec^{-1}1.5)$
25. csc(arccot 5) **26.** $\sec(\operatorname{arccsc}\frac{7}{4})$ **27.** $\cos[\operatorname{arccsc}(-\frac{5}{4})]$ **28.** $\tan[\operatorname{arccsc}(-\frac{5}{3})]$
29. $\tan[\sec^{-1}(-\frac{6}{5})]$ **30.** $\csc[\sec^{-1}(-\frac{8}{7})]$ **31.** $\sin(\csc^{-1}z)$, $z > 0$ **32.** $\cot(\sec^{-1}z)$, $z > 0$
33. $\sec(\cot^{-1}z)$, $z < 0$ **34.** $\tan(\operatorname{arcsec} z)$, $z < 0$ **35.** $\cos(\operatorname{arcsec} 2z)$, $z > 0$ **36.** $\sin(\csc^{-1}3z)$, $z < 0$
37. $\tan[\sec^{-1}(z+1)]$, $z + 1 > 0$ **38.** $\csc[\sec^{-1}(1-z)]$, $1 - z > 0$
39. $\csc\left(\operatorname{arccsc}\frac{3}{z}\right)$, $z > 0$ **40.** $\sec\left(\cot^{-1}\frac{2}{z+1}\right)$, $z + 1 > 0$

Chapter 4 summary

- To show that two functions f and g are inverses of each other
  Show that [1] If $f(x) = y$, then $g(y) = x$, and
  [2] If $g(x) = y$, then $f(y) = x$.
- **Vertical line test for a function** If no vertical line crosses the graph of a relation in more than one place, the relation is a function.
- **Horizontal line test for a one-to-one function** If no horizontal line crosses the graph of a function in more than one place, the function is one to one.
- **Inverse sine function**
  $y = \sin^{-1}x$ means
  1. $\sin y = x$
  2. $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
  3. $-1 \le x \le 1$
- **Inverse cosine function**
  $y = \cos^{-1}x$ means
  1. $\cos y = x$
  2. $0 \le y \le \pi$
  3. $-1 \le x \le 1$
- **Inverse tangent function**
  $y = \tan^{-1}x$ means
  1. $\tan y = x$
  2. $-\frac{\pi}{2} < y < \frac{\pi}{2}$
- **Inverse cotangent function**
  $y = \cot^{-1}x$ means
  1. $\cot y = x$
  2. $0 < y < \pi$
- $\cot^{-1}x = \frac{\pi}{2} - \tan^{-1}x$
- **Inverse secant function** $\sec^{-1}x = \cos^{-1}\frac{1}{x}$ if $|x| \ge 1$
- **Inverse cosecant function** $\csc^{-1}x = \sin^{-1}\frac{1}{x}$ if $|x| \ge 1$

- Summary of the properties of the inverse sine, cosine, and tangent functions.

Function	Domain	Range	Quadrants
$y = \sin^{-1}x$	$\lvert x\rvert \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$	I, IV
$y = \cos^{-1}x$	$\lvert x\rvert \le 1$	$0 \le y \le \pi$	I, II
$y = \tan^{-1}x$	R	$-\frac{\pi}{2} < y < \frac{\pi}{2}$	I, IV

- Summary of the properties of the inverse cotangent, secant, and cosecant functions.

Function	Domain	Range	Quadrants
$y = \cot^{-1}x$	R	$0 < y < \pi$	I, II
$y = \sec^{-1}x$	$\lvert x\rvert \ge 1$	$0 \le y \le \pi, y \ne \frac{\pi}{2}$	I, II
$y = \csc^{-1}x$	$\lvert x\rvert \ge 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}, y \ne 0$	I, IV

Chapter 4 review

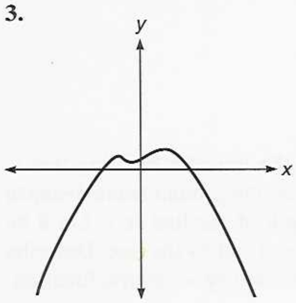
[4–1] Show that the following functions f and g are inverses of each other.

1. $f(x) = 3x - 5;\ g(x) = \dfrac{x+5}{3}$

2. $f(x) = \dfrac{3x}{x+1};\ g(x) = \dfrac{-x}{x-3}$

Each of the following diagrams shows the graph of a function. Use the horizontal line test to determine if the function is or is not one to one.

3.



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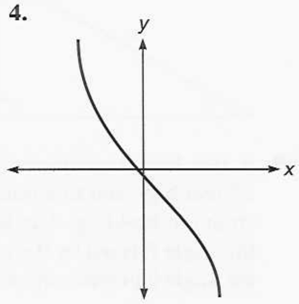


[4–2]

5. Sketch the graph of the inverse sine function.

6. State the domain and range of the inverse sine function.

Find exact values for each of the following expressions in both radians and degrees.

7. $\sin^{-1}(-\frac{1}{2})$

8. $\arcsin\dfrac{\sqrt{3}}{2}$

9. $\sin^{-1}\frac{1}{2}$

10. $\arcsin\left(-\dfrac{\sqrt{2}}{2}\right)$

11. $\arcsin 0$

12. $\sin^{-1}(-1)$

Find approximate values for the following expressions in both radians and degrees.

13. $\sin^{-1}0.9737$

14. $\arcsin(-0.4882)$

Simplify the following expressions.

15. $\cos(\sin^{-1}\frac{3}{5})$

16. $\tan(\sin^{-1}\frac{1}{4})$

17. $\sec[\sin^{-1}(-\frac{2}{3})]$

18. $\tan(\sin^{-1}z),\ z > 0$

19. $\cos[\arcsin(1+z)],\ 1 + z > 0$

[4–3]

20. Sketch the graph of the inverse cosine function.

21. State the domain and range of the inverse tangent function.

Find exact values for each of the following expressions in both radians and degrees.

22. $\cos^{-1}(-\frac{1}{2})$

23. $\arccos\dfrac{\sqrt{3}}{2}$

24. $\cos^{-1}\frac{1}{2}$

25. $\arcsin\left(-\dfrac{\sqrt{2}}{2}\right)$

26. $\arctan\sqrt{3}$

27. $\tan^{-1}(-1)$

Find approximate values for the following expressions in both radians and degrees.

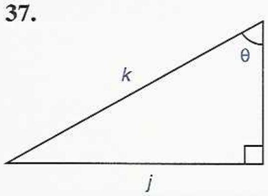
28. $\tan^{-1}1.5601$ **29.** $\arccos 0.4882$
30. $\arccos(-0.3051)$

Simplify the following expressions.

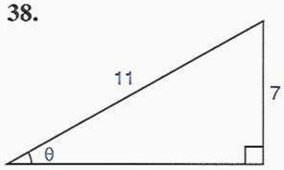
31. $\cos(\tan^{-1}\frac{3}{5})$ **32.** $\sec(\cos^{-1}\frac{1}{4})$
33. $\tan[\cos^{-1}(-\frac{2}{3})]$ **34.** $\cos^{-1}\left(\sin\frac{7\pi}{6}\right)$
35. $\tan(\cos^{-1}z)$, $z > 0$
36. $\cos[\arctan(1 + z)]$, $1 + z > 0$

For the following diagrams state the angle θ in terms of an inverse trigonometric function.

37.



38.



39. A camera is to be placed on the ground x feet from a flagpole that is 35 feet high. Describe the angle determined by the flagpole at the camera in terms of an inverse trigonometric function.

In the following problems describe one value of θ in exact form, in terms of an inverse trigonometric function.

40. $\cos\theta = 0.89$ **41.** $\sin\theta = -0.88$
42. $\frac{1}{2}\sin\theta = -0.1$ **43.** $2\tan\theta = 10$
44. $\sin 2\theta = 0.76$ **45.** $2\cos 3\theta = 1.4$
46. $\cos(2\theta + 3) = 0.6$ **47.** $\sin 2\theta + 3 = 3.6$

[4–4] Compute the exact value of the following expressions in both radians and degrees.

48. $\sec^{-1}2$ **49.** $\operatorname{arccsc}\left(-\frac{2\sqrt{3}}{3}\right)$
50. $\operatorname{arccsc}(-2)$ **51.** $\cot^{-1}(-\sqrt{3})$

Compute approximate values of the following expressions in both radians and degrees.

52. $\csc^{-1}4.3864$ **53.** $\cot^{-1}1.5601$
54. $\operatorname{arcsec}(-4.0420)$ **55.** $\operatorname{arccsc}(-6.6917)$

Simplify the following expressions.

56. $\tan(\csc^{-1}3)$ **57.** $\cot(\operatorname{arcsec} 4)$
58. $\sin(\operatorname{arccot} 5)$ **59.** $\csc[\operatorname{arcsec}(-\frac{7}{4})]$
60. $\sin(\cot^{-1}z)$, $z > 0$ **61.** $\cot(\csc^{-1}z)$, $z > 0$
62. $\tan[\sec^{-1}(z + 1)]$, $z + 1 > 0$
63. $\csc[\sec^{-1}(1 - z)]$, $1 - z > 0$

Chapter 4 test

1. Sketch the graph of the inverse tangent function.
2. State the domain and range of the inverse cosecant function.

Find exact values for each of the following expressions in both radians and degrees.

3. $\sec^{-1}(-2)$ **4.** $\arcsin\frac{\sqrt{2}}{2}$
5. $\csc^{-1}2$ **6.** $\arctan\sqrt{3}$

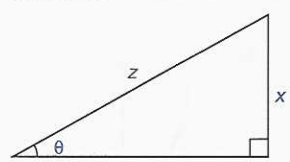
Find approximate values for the following expressions in both radians and degrees.

7. $\tan^{-1}1.4617$ **8.** $\arccos(-0.4169)$
9. $\operatorname{arcsec}(-1.4830)$ **10.** $\operatorname{arccsc} 3.4971$

Simplify the following expressions.

11. $\cot(\sin^{-1}\frac{3}{5})$ **12.** $\sin[\tan^{-1}(-4)]$
13. $\tan[\sec^{-1}(-3)]$ **14.** $\sec(\cos^{-1}z)$, $z > 0$
15. $\sin[\operatorname{arccot}(1 + z)]$, $1 + z > 0$
16. $\tan(\sec^{-1}2z)$, $z > 0$
17. $\sec[\csc^{-1}(1 - z)]$, $1 - z > 0$

18. State the angle θ in terms of an inverse trigonometric function.



19. A taut line is connected to the top of a building that is 52 feet high and to a point on the ground some distance from the building. The length of the line is x. Let θ be the angle formed by the ground and by the line. Describe the angle θ in terms of an inverse trigonometric function.

In the following problems describe one value of θ in exact form in terms of an inverse trigonometric function.

20. $\sec\theta = 2.25$ **21.** $2\sin\theta = -1.88$
22. $\sin 3\theta = 0.75$ **23.** $4\cos(2\theta - 3) = 2.4$

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